Series solutions for odes

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1 Preview

Here are some important series forms of relevant functions.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

You can construct these series by the taylor series of these functions around 0. These functions are analytic i.e their taylor series has an infinite radius of convergence. That means, no matter how big x is, this relation holds well.

2 A new approach

Not every differential equation can be solved exactly with the methods you learnt so far (or infact, any method at all). You might encounter a few where none of the techniques you learnt work. This next approach is about your guess of how the solution looks like and working from that to see how to make your guess better. So, in the spirit of this new approach, let's forget everything we know about solving odes and jump into a problem

Let's solve

$$y'' = x^2$$

We try to solve this by assuming that the solution is of a particular form

$$y = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

This seems like a haphazard guess, but let's try it anyway. then,

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + \dots$$

But we know $y'' = x^2$. So, that means

$$a_2 = a_3 = a_5 = a_6 = \dots = 0$$

and

$$12a_4 = 1$$
$$a_4 = \frac{1}{12}$$

We put all this information back in our guess of y

$$y = a_0 + a_1x + \frac{x^4}{12}$$

This is the same result we get if we had integrated the ode. (a_0 and a_1 are the constants of integration.)

3 Why does this work

We represented y as a sum of the functions $1, x, x^2, x^3, x^4, \dots$. These functions form orthogonal basis for y. For example,

$$x^2 \neq c_0 + c_1x + c_2x^3 + \dots$$

There are no values of the c_is for which the inequality above becomes an equality.

What this means is that the functions are linearly-independent. This makes them very good candidates for solutions to odes.

4 When does this work

Naturally, we cannot use this for every ode. For this to work, the ode must be expressible as

$$y^{\{n\}} + p_{n-1}(x)y^{\{n-1\}} + p_{n-2}(x)y^{\{n-2\}} + \dots = 0$$

where $p_{n-1}(x), p_{n-2}(x), \dots$ are analytic and $y^{\{n\}}$ represent the nth derivative of y.

A function is called analytic if it has an infinite radius of convergence. More usefully, a function is analytic if it doesn't blow up (go to infinity) at finite x.

For example, f(x) = $\frac{1}{x}$ is not analytic because it is infinity at x=0.

A few more examples:

- $y'' + y = 0$

$$y = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + \dots$$

Putting this into our equation, we get

$$2a_2 + a_0 + (6a_3 + a_1)x + (12a_4 + a_2)x^2 + \dots = 0$$

Then, we find

$$a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{6}, a_4 = -\frac{a_2}{12}, a_5 = -\frac{a_3}{20}$$

$$a_2 = -\frac{a_0}{2*1}, a_3 = -\frac{a_1}{3*2}, a_4 = -\frac{a_2}{4*3}, a_5 = -\frac{a_3}{5*4}$$

We see a pattern.

$$a_n = -\frac{a_{n-2}}{n(n-1)} \tag{1}$$

For the even terms,

$$a_2 = -\frac{a_0}{2!}, a_4 = -\frac{a_2}{4*3} = \frac{a_0}{4*3*2} = \frac{a_0}{4!}, a_6 = -\frac{a_4}{6*5} = -\frac{a_0}{6*54*3*2} = -\frac{a_0}{6!}$$

$$a_n = (-1)^{\frac{n}{2}} \frac{a_0}{n!}$$

For the odd terms,

$$a_3 = -\frac{a_1}{3*2}, a_5 = -\frac{a_3}{5*4} = \frac{a_1}{5*4*3*2} = \frac{a_1}{5!}, a_7 = -\frac{a_1}{7!}$$

$$a_n = (-1)^{\frac{n-1}{2}} \frac{a_1}{n!}$$

Rewriting our guess for y,

$$y = a_0 - a_0\frac{x^2}{2!} + a_0\frac{x^4}{4!} + \dots + a_1 x - a_1\frac{x^3}{3!} + a_1\frac{x^5}{5!} + \dots$$

$$y = a_0\big(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\big) + a_1\big(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\big)$$

$$y = a_0 cos(x) + a_1 sin(x)$$

With initial condition $y(0) = 1$, $y'(0) = 3$, we can plug that into our solution to get

$$y = cos(x) + 3sin(x)$$

If you use the sigma (sum) notation, you can find the pattern a lot quicker

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{i=0}^{\infty}(i+2)(i+1)a_{i+2}x^i$$

Replacing i with n because they are arbitary variables,

$$y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' + y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}((n+2)(n+1)a_{n+2} + a_n)x^n = 0$$

From this, we get the same relation as (1) more easily.

- $$y' = 2xy$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{i=0}^{\infty}(i+1)a_{i+1}x^i = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

$$a_1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

$$a_1 + \sum_{n=0}^{\infty}(n+2)a_{n+2}x^{n+1} = \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

So,

$$a_1 = a_3 = a_5 = 0, a_{n+2} = \frac{2a_n}{(n+2)}$$

$$a_0 = a_0, a_2 = a_0, a_4 = \frac{2a_2}{4} = \frac{a_0}{2!}, a_6 = \frac{a_0}{3!}$$

$$y = a_0 + a_0 x^2 + a_0\frac{x^4}{2!} + \frac{x^6}{3!} + \ldots$$

$$y = a_0 \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = a_0 e^{x^2}$$

With initial condition $y(0) = 1$, we can plug that into our solution to get

$$y = e^{x^2}$$

If the sigmas have confused you, try doing this example without the sigmas.

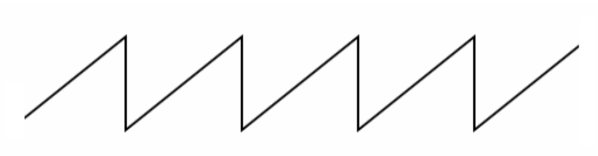
5 A deeper understanding of what a basis is

We previously talked about how the functions $1, x, x^2, x^3, x^4, \ldots$ form a basis for y. That means they are potential candidates to the solution for y.

You also know that an nth order ode has n solutions. But in the previous example, the solution to the ode was an infinite sum of the functions $1, x, x^2, x^3, x^4, \ldots$. Does that mean the ode has infinitely many solutions ?

No ! If you look at the functions $1, x, x^2, x^3, x^4, \ldots$ individually, none of them satisfy the ode. So, no **finite** sum of these could be a solution. However, when it is an infinite sum, that sum converges to two special functions (sine and cosine) which normally wouldn't be represented by a sum of these functions.

So, the thing to remember is that taking an infinite sum might give some 'special' properties. For example. the sawtooth wave below can be written as an infinite sum of sines and cosines. Although sines and cosines, and their finite sums are differentiable, but clearly their infinite sums might not be.



How is this relevant to you? Understanding these properties can help you understand the nature behind many engineering and mathematical situations. However, more relevant to this class, if you applied the concepts to solving an n-th order ode and you found that you had more than n coefficients (the a_ns), then you know that you will have an infinite sum of coefficients and the solution might be something involving a sine or a cosine or an exponential (but not just polynomials).

6 Odes with regular singular points

A point x_0 is a regular singular point of an ode if the ode given by

$$y^{\{n\}} + pn - 1(x)y^{\{n-1\}} + p_{n-2}(x)y^{\{n-2\}} + \ldots + p_0 y = 0$$

has the coefficients in such a way that

$$(x - x_0)^n p_0, (x - x_0)^{n-1} p_1, \ldots, (x - x_0) p_{n-1}$$

are analytic

For example,

- $$x^2y'' + xy' = y$$

  has a regular singular point at 0

  $$y'' + \underbrace{\frac{1}{x}}_{p_1} y' = \underbrace{\frac{1}{x^2}}_{p_0} y$$

  $$x_0 = 0$$

  $$x^2p_0 = -1, xp_1 = 1$$

- $$(x-1)y''' = y$$

  has a regular singular point at 1.

  $$y''' - \underbrace{\frac{1}{x-1}}_{p_0} y = 0$$

  $$x_0 = 1$$

  $$(x-1)^3p_0 = -(x-1)^2 \text{ which is analytic}$$

- However,

  $$x^3y' = y$$

  has an **irregular** singular point at 0.

  $$y' = \frac{1}{x^3}y$$

  $$xp_0 = \frac{1}{x^2} \text{ which still blows up at } x = 0$$

7 Does this change anything ?

Let's try solving a regular singular ode.

$$xy' + y = 0$$

Assuming $y = \sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$, we get

$$a_1x + a_2x^2 + a_3x^3 + \ldots + a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots = 0$$

$$a_0 + 2a_1x + 2a_2x^2 + 2a_3x^3 + \ldots = 0$$

This means all the coefficients have to be zero. Obviously, zero is a solution to the ode. But we want a non-trivial solution. Unfortunately, for regular singular odes, our regular guess is not enough.

You know that the solution to the ode $xy' + y = 0$ is $\frac{1}{x}$. It makes sense that our guess did not work since $\frac{1}{x}$ was not covered by the basis (The solution is not analytic).

So, we guess

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n} = a_0 x^r + a_1 x^{r+1} + \dots$$

Here, r can be negative or even a fraction. We also add a condition.

$$a_0 \neq 0$$

Since r can be arbitary, we want to choose r such that the first term is not zero. That means the first term is a candidate solution to the ode.

8 How do we find r ?

$a_0 x^r$ is a candidate solution. Let's plug it into our ode.

$$x a_0 r x^{r-1} + a_0 x^r = 0$$
$$r a_0 x^r + a_0 x^r = 0$$
$$r + 1 = 0$$
$$r = -1$$

(If you aren't convinced by this, try plugging $\sum_{n=0}^{\infty} a_n x^{n+r}$ into the ode and you will get the same result (with more messy terms)). So, our guess should be $\sum_{n=0}^{\infty} a_n x^{n-1}$. Plugging this in, you will find that all the other terms are zero.

A real example

$$x^2 y'' - 4xy' + 4y = 0$$

We first check to see if it is singular. It is. So, we make this guess

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = a_0 x^r + a_1 x^{r+1} + a_2 x_{r+2} + \dots$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} = r a_0 x^{r-1} + (r+1) a_1 x^r + (r+2) a_2 x_{r+1} + \dots$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} = r(r-1) a_0 x^{r-2} + (r+1) r a_1 x^{r-1} + (r+2)(r+1) a_2 x_r + \dots$$

Plugging these into the ode and looking at the x^r terms

$$r(r-1) a_0 x^r - 4 r a_0 x^r + 4 a_0 x^r = 0$$

$$r(r-1) - 4r + 4 = 0$$

$$r^2 - 5r + 4 = 0$$
$$r = 4, 1$$

Since a second order ode has 2 solutions, the solution is

$$y = c_1x + c_2x^4$$

With initial condition $y(1) = 4$, $y(-1) = 2$, we can plug that into our solution to get

$$y = x + 3x^4$$

This is similar to what you did for second order constant-coefficient linear odes.

9 An important remark

Here, we have looked at the cases when the values of r are distinct and the solutions are finite sums. This is not always the case. If both the values of r are the same, we need to look for a new linearly independent solution. Also, if the solutions are infinite sums, r = 4 and r = 1 are the same because

$$y_1 = \ldots + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$$

is not linearly independent from

$$y_2 = \ldots + a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$$

But discussing those cases is slightly advanced at this stage.

10 Irregular singular odes

$$x^3y'' = y$$

is an irregular singular equation.

Trying our latest guess, we plug in a_0x^r

$$r(r-1)a_0x^{r+1} = a_0x^r$$
$$r(r-1)a_0x = a_0$$

The only way this is true for all x is if a_0 is zero. However, that disagrees with our very first assumption ($a_0 \neq 0$).

So, irregular singular odes are much harder to solve.